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Further Development of Baranoff's Analytical Form of the Kernel of the Generalized Possio's Integral Equation

Übersicht: Die analytische Form der Kernfunktion, die in der verallgemeinerten Possioschen Integralgleichung für den Fall eines in Unterschallströmung harmonisch schwingenden dünnen Tragflügels auftritt, wird weiterentwickelt. Die Beziehung zu anderen Formen wird dargestellt. Bestimmte parametrische Beziehungen, die bei nicht ebenen Tragflügeln verwendet werden könnten, werden gegeben.

Summary: The completely analytical form of the kernel function, appearing in the generalized Possio integral equation for the case of the thin wing oscillating harmonically in subsonic flow, is further developed. The relationship to other presented forms is brought out. Certain parametric relationships which could be used in the case of non-planar wings are given.

Résumé: On a fait avancer le développement de la fonction de noyau en forme entièrement analytique telle qu'elle se produit dans l'équation de Possio généralisée pour le cas de l'aile mince oscillant harmoniquement en écoulement subsonique. La relation avec autres formes est démontrée. On présente certaines relations paramétriques qui puissent être utilisées en cas des ailes portantes non-planes.

1. Introduction

A. von Baranoff [1] has given for the first time a completely analytical form of the kernel function which appears in the generalized *Possio's* integral equation. (This equation relates the downwash to the pressure-doublet strength at the surface of a thin wing oscillating harmonically in subsonic flow.) His development discredited previous claims that it was not possible to obtain such a form. (Refer to [1] for a discussion of this matter.) In view of the fundamental role played by the kernel function in the formulation and development of numerical lifting surface theories, the further development of the new analytical form should be carried out. The present paper therefore extends the work of *von Baranoff* and discusses some directions for still further development.

2. Baranoff's Analytical Form of the Kernel

A. von Baranoff [1] gives an ingeniously derived analytical form of the kernel of the generalized *Possio* integral equation. His results are the following. With the abbreviation

$$(1)\quad \begin{cases} x - x' = x_0\,, \\ y - y' = y_0\,, \end{cases}$$

where x' and y' are the integration variables, the kernel reads:

$$(2)\quad K(x_0, y_0) = e^{-ikx_0} \lim_{z\to 0} \frac{\partial^2}{\partial z^2} \int_q^\infty \frac{e^{-ik\lambda/\beta^2}}{\sqrt{\lambda^2+\beta^2 r^2}}\, d\lambda\,,$$

where

$$r = \beta\sqrt{y_0^2+z^2}\,,$$
$$q = M\sqrt{x_0^2+r^2} - x_0\,,$$
$$k = \omega\, l/U\,,$$

U = stream speed, ω = oscillation frequency,

$\beta = \sqrt{1-M^2}$, M = *Mach* number, l = reference length.

Thereupon *von Baranoff* obtains a set of coefficients

$$S_i(p) \qquad (i = 0, 1, 2, \cdots, \infty)\,,$$

in terms of the parameter p, for which he explicitly exhibits the first seven. The first three are:

$$(3)\quad \begin{cases} S_0(p) = \sqrt{1+p^2} - p\,, \\ S_1(p) = p\sqrt{1+p^2} - \dfrac{1}{2} - p^2 + \log\dfrac{p+\sqrt{1+p^2}}{2p}\,, \\ S_2(p) = \left(\dfrac{4}{3} - \dfrac{2}{3}p^2\right)\sqrt{1+p^2} - p + \\ \qquad\qquad + \dfrac{2}{3}p^3 - 2p\log\dfrac{p+\sqrt{1+p^2}}{2p}\,. \end{cases}$$

(Here the typographical error in the third term of S_2 in [1] is corrected to read p^3 instead of p^2.) From the paper (translated):

"With

$$(15)\quad \begin{cases} I(p,s) = i\, e^{ips} I_1(s)\,[\mathrm{Ci}(ps) - i\,\mathrm{si}(ps)] + \\ + \displaystyle\sum_{\nu=0}^{\infty}\left(\frac{s}{2}\right)^{2\nu} S_{2\nu}(p) + i\sum_{\nu=0}^{\infty}\left(\frac{s}{2}\right)^{2\nu+1} S_{2\nu+1}(p)\,, \end{cases}$$

a function defined for all values s and all $0 < p < \infty$, the final result reads

for $x_0 < M|y_0|$:

$$(16)\quad \begin{cases} K(x_0,y_0) = e^{-ik(x_0+q_0/\beta^2)} \times \\ \qquad \times\left[-\dfrac{\sqrt{x_0^2+\beta^2 y_0^2}+x_0}{y_0^2\sqrt{x_0^2+\beta^2 y_0^2}} + \dfrac{ik}{|y_0|} I(p,s)\right] \end{cases}$$

and for $x_0 > M|y_0|$:

$$(17)\quad \begin{cases} K(x_0,y_0) = \\ = -\dfrac{2k}{|y_0|} e^{-ikx_0} K_1(k|y_0|) + e^{ik(x_0+q_0/\beta^2)} \times \\ \qquad \times\left[\dfrac{\sqrt{x_0^2+\beta^2 y_0^2}-x_0}{y_0^2\sqrt{x_0^2+\beta^2 y_0^2}} + \dfrac{ik}{|y_0|}\bar{I}(p,s)\right], \end{cases}$$

where the overscored $I(p,s)$ signifies the conjugate complex value."

Here q_0 is the value of q at the surface. The condition $x_0 < M|y_0|$ is the same as $q_0 > 0$, and the condition $x_0 > M|y_0|$ is the same as $q_0 < 0$.

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Note that *Landahl*'s [2] parameter u_1 is

$$u_1 = \frac{1}{\beta}\left(M\sqrt{1+\left(\frac{x_0}{\beta r_1}\right)^2} - \frac{x_0}{\beta r_1}\right)$$

and for the planar case where $r_1 = |y_0|$ we see that

$$p = |u_1|\,,$$
$$s \equiv k\,|y_0| = k_1\,,$$
$$q = M R - x_0 = \beta r u_1 = \beta^2 r_1 u_1\,.$$

With regard to the series in the expression for $I(p, s)$, since $s = k\,|y_0|$, we have in effect an expansion in s (i.e., k_1): it is essentially a frequency expansion. It should be of interest to compare this expansion with that of *Watkins* et al. [3].

Following step by step *von Baranoff*'s paper, after some labor one verifies the analysis with but a single exception which *von Baranoff* was kind enough to clarify (private communication). In order to rigorously establish the validity of the results, one must demonstrate that, in fact, for all $0 < p < \infty$, the series in (15) involving the function-coefficients $S_{2\nu+1}(p)$ and $S_{2\nu}(p)$ do converge. This follows readily for $p \geqq 1$. For $p < 1$ *von Baranoff* states:

"... The next question is the convergence of the power series (11) in s, where the functions are defined by (13) and (12), where the latter relations, though established by inspection from the series (10), are valid throughout the whole domain of p because of the analytic character of the *Laplace* integral (8). We observe (as a matter of fact this is based here on numerical calculations only; a proof would be valuable) that they are positive and monotonically decreasing. The even functions are finite at $p = 0$, the odd possess a logarithmic infinity at $p = 0$."

Apparently he has no rigorous mathematical proof of the convergence. (I have also tried to construct such a proof, but without success.) However, he does accomplish majorization of the even series if $p = 0$, and establishes the convergence of the odd series in the case where p approaches zero but remains finite. And for $p \geqq 1$ it is easy to prove that the functions $S_{2\nu}$ and $S_{2\nu+1}$ are both positive and monotonically decreasing, and hence both series converge there. Thus it seems entirely reasonable that since the series converge for very small (0 or finite) p and for $p \geqq 1$, the range in between should not be exceptional, and therefore one expects to find convergence in the full range $0 < p < \infty$, even though a complete proof is not in hand.

Von Baranoff gives (private communication) further a helpful supplement to his paper which one easily verifies and which reads: "In order to establish the law of dependence on the parameter ν in the case of the even functions $S_{2\nu}(p)$ we proceed as follows.

Let be $p = 0$. Then the integral (8) is written

$$I(0, s) = \int_0^\infty \frac{J_1(t)\,dt}{t + i s}\,.$$

Its real part is

$$\int_0^\infty \frac{t\,J_1(t)\,dt}{t^2 + s^2}$$

which according to (9) and (11) is equal to

$$-\frac{\pi}{2} I_1(s) + \sum_{\nu=0}^{\infty}\left(\frac{s}{2}\right)^{2\nu} S_{2\nu}(0)\,.$$

Watson (*Bessel* Functions, p. 425) gives the result

$$\int_0^\infty \frac{J_0(a x)}{x^2 + k^2}\,dx = \frac{\pi}{2k}\left[I_0(a k) - L_0(a k)\right]$$

which when differentiated with regard to the parameter a gives (after certain manipulation) finally the desired relation

$$S_{2\nu}(0) = \frac{\pi/2}{\Gamma(\nu + 1/2)\,\Gamma(\nu + 3/2)}\,.$$

Because of

$$0 < S_{2\nu}(p) \leqq S_{2\nu}(0)$$

the even functions are seen to decrease with ν at least as

$$\frac{1}{\Gamma(\nu + 1/2)\,\Gamma(\nu + 3/2)}\,.$$

The convergence of the power series

$$\sum_{\nu=0}^{\infty}\left(\frac{s}{2}\right)^{2\nu} S_{2\nu}(p)$$

is thus established.

The case of the odd functions $S_{2\nu+1}(p)$ is a little bit more involved because of the logarithmic term. The argument runs however in a similar way. The imaginary part of $I(0, s)$ gives (the relation is known as *Sonine*'s integral)

$$-s\int_0^\infty \frac{J_1(t)\,dt}{t^2 + s^2} = K_1(s) - \frac{1}{s}\,.$$

We deduce then easily that for small values of p the odd functions behave as

$$S_{2\nu+1}(p) \sim -\frac{\log(2p)}{\nu!\,(\nu+1)!} - \frac{\Psi(\nu+1) + C + \Psi(\nu+2)}{2\,\nu!\,(\nu+1)!} + C$$

(C = *Euler*'s constant). The convergence of the series

$$\sum_{\nu=0}^{\infty}\left(\frac{s}{2}\right)^{2\nu+1} S_{2\nu+1}(p)$$

for $0 < p < \infty$ is then equally demonstrated."

Introduce the following notation for the odd factorial:

$$n!! = 1\cdot 3\cdot 5 \cdots (2n-1)\,,\quad 0!! = 1\,.$$

As a matter of possible future interest, with considerable additional labor one obtains general expressions for the coefficient-functions as follows:

$$\left|\begin{aligned} S_{2\nu} = \sqrt{1+p^2}\,\Bigg\{ & \frac{4^\nu}{\nu!!\,(\nu+1)!!} + \\ & + \sum_{r=1}^{\nu}(-2p^2)^r\Bigg[\frac{4^{\nu-r}\,r!!}{\nu!!\,(\nu+1)!!\,r!} + \\ & + \sum_{l=1}^{r}\frac{1}{l!!}\Bigg(\frac{4^{\nu-r}\,(r-l)!!}{(\nu-l+1)!!\,(\nu-l)!!\,l!\,(r-l)!} - \\ & - \frac{(r-l)!}{(r-l+1)!!\,(\nu-l)!\,(\nu-l+1)!\,(l-1)!}\Bigg)\Bigg]\Bigg\} + \end{aligned}\right. \tag{4}$$

$$
\begin{aligned}
&+ p\,\frac{2\nu\left(\sum\limits_{2}^{\nu}\frac{1}{m}\right)-1}{\nu\,!\,\nu\,!} + p \sum_{r=1}^{\nu-1} \frac{(-2p^2)^r}{(\nu-1-r)\,!\,(\nu-r)\,!} \times \\
&\times \left[\frac{2(\nu-r)(2r+1)\left(\sum\limits_{1}^{\nu-r}\frac{1}{m}\right)-2\nu-1}{(r+1)\,!!\,r\,!\,(2r+1)(\nu-r)} + \right. \\
&\left. + \sum_{l=1}^{r} \frac{2r-4l+1}{l\,!!\,(r-l+1)\,!!\,(r-l)\,!!\,l\,!\,(2r-2l+1)\,l} \right] - \\
&- p\,\frac{(-2p^2)^\nu}{(\nu+1)\,!!\,\nu\,!} - 2p\log\left(\frac{\sqrt{1+p^2}+p}{2p}\right) \times \\
&\times \sum_{r=0}^{\nu-1} \frac{(-2p^2)^r}{(r+1)\,!!\,(\nu-r-1)\,!\,(\nu-r)\,!\,r\,!} \\
&\qquad\qquad \text{for } \nu \geqq 2,
\end{aligned} \tag{4}
$$

$$
S_0 = \sqrt{1+p^2} - p, \tag{5}
$$

$$
\begin{aligned}
S_2 &= \left(\frac{4}{3} - \frac{2}{3}p^2\right)\sqrt{1+p^2} - p + \\
&\quad + \frac{2}{3}p^3 - 2p\log\left(\frac{p+\sqrt{1+p^2}}{2p}\right)
\end{aligned} \tag{6}
$$

and

$$
\begin{aligned}
S_{2\nu+1} &= p\sqrt{1+p^2}\sum_{r=0}^{\nu}(-2p^2)^r \times \\
&\times \sum_{l=0}^{r}\frac{1}{l\,!}\left[\frac{2\cdot 4^{\nu-r}(r-l)\,!!}{(\nu-l)\,!!\,(\nu-l+1)\,!!\,(l+1)\,!!} - \right. \\
&\left. - \frac{(r-l)\,!}{(r-l+1)\,!!\,l\,!!\,(\nu-l)\,!\,(\nu-l+1)\,!}\right] - \\
&- \frac{2(\nu+1)\left(\sum\limits_{1}^{\nu+1}\frac{1}{m}\right)-1}{2(\nu+1)\,!\,(\nu+1)\,!} - \\
&- \frac{1}{2}\sum_{r=1}^{\nu}\frac{(-2p^2)^r}{(\nu-r)\,!\,(\nu-r+1)\,!} \times \\
&\times \left[\frac{2(\nu-r+1)\left(\sum\limits_{r}^{\nu-r+1}\frac{1}{m}\right)-1}{r\,!!\,r\,!\,(\nu-r+1)} + \right. \\
&\left. + \sum_{l=1}^{r}\frac{(2r-2l+1)(2l-1)-4l^2}{(r-l+1)\,!!\,l\,!!\,(r-l)\,!\,l\,!\,(2l-1)\,l}\right] + \\
&+ \frac{(-2p^2)^{\nu+1}}{2(\nu+1)\,!!\,(\nu+1)\,!} + \log\left(\frac{p+\sqrt{1+p^2}}{2p}\right) \times \\
&\times \sum_{r=0}^{\nu}\frac{(-2p^2)^r}{r\,!!\,r\,!\,(\nu-r)\,!\,(\nu-r+1)\,!} \\
&\qquad\qquad \text{for } \nu \geqq 1,
\end{aligned} \tag{7}
$$

$$
S_1 = p\sqrt{1+p^2} - \frac{1}{2} - p^2 + \log\left(\frac{p+\sqrt{1+p^2}}{2p}\right). \tag{8}
$$

Finally, the kernel function K_1 (see e. g. *Landahl* [2]) may be developed from *von Baranoff*'s analytical form as follows:

a) $q_0 > 0$:

$$
K(x_0, y_0) = \frac{e^{-ikx_0}}{r_1^2}\,e^{-isp}\left[is\,I(p,s) - \frac{R+x_0}{R}\right], \tag{9}
$$

$$
\begin{aligned}
\frac{R+x_0}{R} &= 1 - \frac{p}{\sqrt{1+p^2}} + \\
&\quad + \frac{M}{\left(1 - M\dfrac{p}{\sqrt{1+p^2}}\right)(1+p^2)} = \\
&= (1+M)\,\frac{1-p/\sqrt{1+p^2}}{1-M\,p/\sqrt{1+p^2}},
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
I(p,s) &= i\,e^{ips}\,I_1(s)\,[\mathrm{Ci}(ps) - i\,\mathrm{si}(ps)] + \\
&\quad + \sum_{\nu=0}^{\infty}\left(\frac{s}{2}\right)^{2\nu}\left[S_{2\nu}(p) + i\,\frac{s}{2}\,S_{2\nu+1}(p)\right],
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
K(x_0, y_0) &= \\
&= -\frac{e^{-ikx_0}}{r_1^2}\left\{ s\,I_1(s)\,[\mathrm{Ci}(ps) - i\,\mathrm{si}(ps)] + \right. \\
&\quad + 2\,e^{-isp}\sum_{\nu=0}^{\infty}\left(\frac{s}{2}\right)^{2\nu+1}\left[\frac{s}{2}S_{2\nu+1}(p) - i\,S_{2\nu}(p)\right] + \\
&\quad \left. + e^{-isp}(1+M)\,\frac{1-p/\sqrt{1+p^2}}{1-M\,p/\sqrt{1+p^2}}\right\};
\end{aligned} \tag{12}
$$

b) $q_0 < 0$:

$$
\begin{aligned}
K(x_0, y_0) &= \frac{e^{-ikx_0}}{r_1^2}\left\{ -2s\,K_1(s) + \right. \\
&\quad \left. + e^{-isp}\left[is\,\bar{I}(p,s) + \frac{R-x_0}{R}\right]\right\},
\end{aligned} \tag{13}
$$

$$
\frac{R-x_0}{R} = (1-M)\,\frac{1+p/\sqrt{1+p^2}}{1-M\,p/\sqrt{1+p^2}}, \tag{14}
$$

$$
\begin{aligned}
\bar{I}(p,s) &= -i\,e^{-ips}\,I_1(s)\,[\mathrm{Ci}(ps) + i\,\mathrm{si}(ps)] + \\
&\quad + \sum_{\nu=0}^{\infty}\left(\frac{s}{2}\right)^{2\nu}\left[S_{2\nu}(p) - i\,\frac{s}{2}\,S_{2\nu+1}(p)\right],
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
K(x_0, y_0) &= \frac{e^{-ikx_0}}{r_1^2}\left\{ -2s\,K_1(s) + \right. \\
&\quad + e^{-i2ps}\,s\,I_1(s)\,[\mathrm{Ci}(ps) + i\,\mathrm{si}(ps)] + \\
&\quad + 2\,e^{-isp}\sum_{\nu=0}^{\infty}\left(\frac{s}{2}\right)^{2\nu+1} \times \\
&\quad \times \left[\frac{s}{2}S_{2\nu+1}(p) + i\,S_{2\nu}(p)\right] + \\
&\quad \left. + e^{-isp}(1-M)\,\frac{1+p/\sqrt{1+p^2}}{1-M\,p/\sqrt{1+p^2}}\right\}.
\end{aligned} \tag{16}
$$

3. Parametric Relationships of the Kernel-Function Integral

Consider the integral form corresponding to the results of *Landahl* [2]:

$$
I_{p,m}(u,k) = \int^{u}\frac{x^p\,e^{-ikx}}{(1+x^2)^{m+1/2}}\,dx, \tag{17}
$$

then

$$
I_{p+1,m} = i\left(\frac{\partial I_{p,m}}{\partial k}\right)_{u=\mathrm{const}} \tag{18}
$$

and integrating both sides between 0 and k:

$$
I_{p,m} = J_{p,m} - i\int_0^k I_{p+1,m}(u,k')\,dk', \tag{19}
$$

with

$$J_{p,m} = \int^{u} \frac{x^p \, dx}{(1+x^2)^{m+1/2}} \quad (p, m = 0, 1, \ldots)\,. \tag{20}$$

For the special case where $p = 0$, define

$$I_m(u_1, k) = \int_{u_1}^{\infty} \frac{e^{-iku} \, du}{(1+u^2)^{m+1/2}}\,, \tag{21}$$

in terms of which the following relationships are true:

$$I_m = -\frac{i}{k} \frac{e^{-iku_1}}{(1+u_1^2)^{m+1/2}} - \frac{2m+1}{k} \frac{\partial I_{m+1}}{\partial k}\,, \tag{22}$$

$$\left\{ \begin{aligned} I_{m+1} = & \int_{u_1}^{\infty} \frac{du}{(1+u^2)^{m+3/2}} - \\ & - \frac{1 - e^{-iku_1}}{(2m+1)\, u_1 (1+u_1^2)^{m+1/2}} - \\ & - \frac{1}{2m+1} \int_0^k k' \, I_m(u_1, k') \, dk'\,. \end{aligned} \right. \tag{23}$$

The latter becomes for $m = 0$ and 1:

$$\left\{ \begin{aligned} I_1 = 1 - & \frac{u_1}{\sqrt{1+u_1^2}} - \frac{1 - e^{-iku_1}}{u_1 \sqrt{1+u_1^2}} - \\ & - \int_0^k k' \, I_0(u_1, k') \, dk'\,, \end{aligned} \right. \tag{24}$$

$$\left\{ \begin{aligned} I_2 = & \frac{2}{3}\left(1 - \frac{u_1}{\sqrt{1+u_1^2}}\right) - \frac{1}{3} \frac{u_1}{(1+u_1^2)^{3/2}} - \\ & - \frac{1 - e^{-iku_1}}{3\, u_1 (1+u_1^2)^{3/2}} - \frac{1}{3} \int_0^{-k} k' \, I_1(u_1, k') \, dk'\,. \end{aligned} \right. \tag{25}$$

One might consider this last relationship, Eq. (25), in connection with *von Baranoff*'s analytical form, as a practical means of obtaining the corresponding analytical form for the non-planar case.

4. References

[1] *A. von Baranoff:* Über die analytische Form des Kerns der verallgemeinerten Possioschen Integralgleichung. Z. Flugwiss. 14 (1966), pp. 461–464.

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[3] *C. E. Watkins, H. L. Runyan, D. S. Woolston:* On the kernel of the integral equation relating the lift and downwash distributions of oscillating finite wings in subsonic flow. NACA Rep. 1234 (1953).

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